The Spot Rate Curve

Gary Schurman, MBE, CFA

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The spot rate is calculated by finding the discount rate that makes the present value of a zero-coupon bond equal to its price. In this white paper we will construct an approximation of the ten year spot rate curve using US Treasury Strip price quotes. To that end we will use the following market data...

US Treasury Strip Market Data

The tables below present dollar market prices and relevant discount factors at time zero of three different $100 US Treasury Strips maturing at three different dates along the market spot rate curve...

Table 1: Market Prices at 02-02-2019

Date	Time	Bid	Ask	Average
02/02/19	0.00	100.000	100.000	100.000
11/15/24	5.79	86.414	86.463	86.439
02/15/29	10.04	76.248	76.323	76.286

Table 2: Market Discount Factors at 02-02-2019

Symbol	Factor
D_1	1.00000
D_2	0.86439
D_3	0.76286

Notes: The discount factors in Table 2 above are defined as the average of the bid and ask price divided by $100. The first table row is the price at time zero ($100: no discount), the second table row is the price at the approximate middle of the curve, and the third table row is the price is at the end of the curve.

Building The Model

We will define the variable P_0 to be the time zero price of a treasury strip that matures at time t, the variable C_t to be cash flow (i.e. stripped principal and/or stripped interest) received at time t, and the variable D_t to be the discount factor over the time interval $[0, t]$. The equation for the price of a treasury strip at time zero is...

$$P_0 = C_t \, D_t \;\; \text{...such that...} \;\; D_t = \frac{P_0}{C_t} \tag{1}$$

We will define the variable r_t to be the continuous-time spot rate at time t. The equation for the discount factor D_t in Equation (1) above as a function of the spot rate at that time is...

$$D_t = \text{Exp}\left\{ - r_t \, t \right\} \;\; \text{...such that...} \;\; r_t = -\frac{\ln(D_t)}{t} \tag{2}$$

We will define the variable $\bar{r}_t$ to be the discrete-time spot rate at time t. Using Equation (2) above the equation for the discrete-time spot rate given the continuous-time rate is...

$$\bar{r}_t = \text{Exp}\left\{ r_t \right\} - 1 \tag{3}$$

We want to model the discount factor in Equation (2) above via the following parabola equation...

$$D_t = a \, t^2 + b \, t + c \;\; \text{...such that...} \;\; \frac{\delta D_t}{\delta t} = 2 \, a \, t + b \tag{4}$$

We want to build an approximation of the market spot rate curve using parabola Equation (4) above. To that end we will define the following matrix and vectors...

$$\mathbf{A} = \begin{bmatrix} t_1^2 & t_1^1 & t_1^0 \\ t_2^2 & t_2^1 & t_2^0 \\ t_3^2 & t_3^1 & t_3^0 \end{bmatrix} \text{ ...and... } \vec{\mathbf{u}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ ...and... } \vec{\mathbf{v}} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \text{ ...such that... } \mathbf{A}\,\vec{\mathbf{u}} = \vec{\mathbf{v}} \tag{5}$$

Using market data in Tables 1 and 2 above we can redefine the matrix and vectors in Equation (5) above as...

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 33.48 & 5.79 & 1 \\ 100.72 & 10.04 & 1 \end{bmatrix} \text{ ...and... } \vec{\mathbf{u}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ ...and... } \vec{\mathbf{v}} = \begin{bmatrix} 1.00000 \\ 0.86439 \\ 0.76286 \end{bmatrix} \text{ ...such that... } \mathbf{A}\,\vec{\mathbf{u}} = \vec{\mathbf{v}} \tag{6}$$

Using the matrix and vector definitions in Equation (6) above we can solve for our parabola parameters as follows...

$$\mathbf{A}^{-1}\,\vec{\mathbf{v}} = \vec{\mathbf{u}} = \begin{bmatrix} -0.00004 \\ -0.02318 \\ 1.00000 \end{bmatrix} \tag{7}$$

Complication - The Spot Rate At Time Zero

Note that using Equation (2) above the continuous-time spot rate at time zero is...

$$r_0 = -\frac{\ln(1.00)}{0} = \frac{0}{0} \text{ ...which is undefined} \tag{8}$$

Since in Equation (8) above we have zero divided by zero then we can use L'Hopital's Rule to solve that equation. Note that using Equation (4) above the derivative of the numerator and denominator of Equation (2) above with respect to time are...

$$\frac{\delta}{\delta t} - \ln(D_t) = -2\,a\,t - b \text{ ...and... } \frac{\delta}{\delta t}\,t = 1 \tag{9}$$

Using the derivatives in Equation (9) above and L'Hopital's Rule the solution to Equation (8) above is...

$$\lim_{t \to \infty} \frac{-2\,a\,t - b}{1} = -b \tag{10}$$

The Market Spot Rate Curve

The table below presents the discount factor, continuous-time spot rate, and discrete-time spot rate at the end of each year over a ten year period...

Year	DFactor	CT Rate	DT Rate
0.00	1.0000	0.02318	0.02345
1.00	0.9768	0.02350	0.02377
2.00	0.9535	0.02383	0.02411
3.00	0.9301	0.02417	0.02446
4.00	0.9066	0.02452	0.02483
5.00	0.8830	0.02489	0.02520
6.00	0.8593	0.02527	0.02559
7.00	0.8356	0.02567	0.02600
8.00	0.8117	0.02608	0.02642
9.00	0.7878	0.02651	0.02686
10.00	0.7637	0.02695	0.02732

Example: The calculations for year three are...

Using Equations (4) and (7) above the discount factor at the end of year three is...

$$\text{Discount Factor} = -0.00004 \times 3^2 - 0.02318 \times 3 + 1 = 0.9301 \tag{11}$$

Using Equations (2) and (11) above the continuous-time spot rate at the end of year three is...

$$\text{CT Rate} = -\frac{\ln(0.9301)}{3} = 0.02417 \tag{12}$$

Using Equations (3) and (12) above the discrete-time spot rate at the end of year three is...

$$\text{DT Rate} = \text{Exp}\left\{0.02417\right\} - 1 = 0.02446 \tag{13}$$

The graph of the discrete-time spot rate curve over the time interval $[0, 10]$ is...